

# The coloring problem for $\{P_5, \overline{P_5}\}$ -free graphs and $\{P_5, K_p - e\}$ -free graphs is polynomial

D.S. Malyshev\* and O.O. Lobanova†

## Abstract

We show that determining the chromatic number of a  $\{P_5, \overline{P_5}\}$ -free graph or a  $\{P_5, K_p - e\}$ -free graph can be done in polynomial time.

Keywords: computational complexity, coloring problem, hereditary class, efficient algorithm

## 1 Introduction

A *coloring* is an arbitrary mapping from the set of vertices or edges of a graph into a set of colors of the graph such that any adjacent vertices (or edges) are colored with different colors. The minimal number of colors sufficient for coloring a graph  $G$  is said to be the *chromatic number* of  $G$  denoted by  $\chi(G)$ . The *coloring problem* is to decide whether  $\chi(G) \leq k$  or not for given graph  $G$  and a number  $k$ . A similar *k-colorability problem* is to check whether a given graph can be colored with at most  $k$  colors. Both problems can be naturally defined in another way via partition into independent sets. An *independent set* of graph is an arbitrary set of pairwise nonadjacent vertices. A coloring is partitioning of vertex set of a graph into independent subsets called *color classes*.

---

\*National Research University Higher School of Economics, 25/12 Bolshaja Pecherskaja Ulitsa, Nizhny Novgorod, 603155, Russia; Lobachevsky State University of Nizhny Novgorod, 23 Gagarina Avenue, Nizhny Novgorod, 603950, Russia; Email: dsmalyshev@rambler.ru

†Lobachevsky State University of Nizhny Novgorod, 23 Gagarina Avenue, Nizhny Novgorod, 603950, Russia; Email: olga-olegov@yandex.ru

There is a natural lower bound for the chromatic number of a graph. A *clique* in a graph is a subset of pairwise adjacent vertices. The size of a maximum clique in a graph  $G$  is called the *clique number* of  $G$  denoted by  $\omega(G)$ . Clearly,  $\chi(G) \geq \omega(G)$ . Sometimes, computing  $\omega(G)$  helps to determine  $\chi(G)$  [8, 14].

A class of graphs is called *hereditary* if it is closed under isomorphism and deletion of vertices. It is well known that any hereditary (and only hereditary) graph class  $\mathcal{X}$  can be defined by a set of its forbidden induced subgraphs  $\mathcal{S}$ . We write  $\mathcal{X} = \text{Free}(\mathcal{S})$  in this case, and graphs in  $\mathcal{X}$  are said to be  $\mathcal{S}$ -free. If  $\mathcal{S} = \{G\}$ , then we write " $G$ -free" instead of " $\{G\}$ -free".

We say that  $\mathcal{X}$  is *easy* for the coloring problem if  $\mathcal{X}$  is hereditary and the problem can be polynomially solved for it.

The computational complexity of the coloring problem was completely determined for all classes of the form  $\text{Free}(\{G\})$  [11]. A study of forbidden pairs was also initiated in [11]. A complete complexity dichotomy appeared hard to obtain even in the cases of two four-vertex and connected five-vertex forbidden induced subgraphs [12, 13]. For all but three cases either NP-completeness or polynomial-time solvability was shown in the family of hereditary classes defined by four-vertex forbidden induced structures [12]. The remaining three classes are stubborn. A similar result was obtained in [13] for two connected five-vertex forbidden induced fragments, where the number of open cases was 13. Recently, it was reduced to 11 [14]. We reduce the number to nine by showing that the coloring problem can be solved for  $\{P_5, \overline{P_5}\}$ -free and  $\{P_5, K_p - e\}$ -free graphs in polynomial time.

## 2 Notation

As usual,  $P_n, C_n, O_n, K_n$  stand respectively for the simple path, the chordless cycle, the empty graph, the complete graph with  $n$  vertices respectively. A graph  $K_p - e$  is obtained from  $K_p$  by deleting an arbitrary edge. A formula  $N(x)$  means the neighborhood of a vertex  $x$  of some graph. For a graph  $G$  and a set  $V' \subseteq V(G)$ ,  $G(V')$  denotes the subgraph of  $G$  induced by  $V'$ .

We refer to textbooks in graph theory for any graph terminology undefined here.

### 3 Auxilliary results

#### 3.1 Decomposition by clique separators and its applications to the coloring problem

A *clique separator* in a graph is a clique whose removal increases the number of connected components. For example, the graph  $K_p - e$  has a clique separator with  $p - 2$  vertices. If a graph  $G$  has a clique separator  $Q$ , then  $V(G) \setminus Q$  can be arbitrarily partitioned into nonempty subsets  $A$  and  $B$  such that no vertex of  $A$  is adjacent to a vertex of  $B$ . Let  $G_1 \triangleq G(A \cup Q)$  and  $G_2 \triangleq G(B \cup Q)$ . We repeat a similar decomposition until no further decomposition is possible. The whole process can be represented by a binary decomposition tree whose leaves correspond to some induced subgraphs of  $G$  without clique separators. There exists an  $O(mn)$ -time algorithm for constructing some binary decomposition tree for any graph with  $n$  vertices and  $m$  edges [15].

**Lemma 1** *For each graph  $G$ ,  $\chi(G) = \max(\chi(G_1), \chi(G_2))$ .*

**Proof.** Without loss of generality,  $\chi(G_1) \leq \chi(G_2)$ . Let us consider a partial coloring of  $G$  induced by an optimal coloring of  $G_2$  and color classes of  $G_1$  in its optimal coloring containing all vertices of  $Q$ . These color classes can be colored with colors assigned to elements of  $Q$  in the partial coloring. To color the remaining part of  $A$ , it is enough  $\chi(G_1) - |Q|$  colors distinct to the colors of  $Q$ . The set  $B$  has  $\chi(G_2) - |Q| \geq \chi(G_1) - |Q|$  color classes with colors of this type. Hence,  $G$  can be colored with  $\chi(G_2)$  colors. So,  $\chi(G) = \chi(G_2)$ . ■

A maximal induced subgraph of a given graph without proper clique separators will be called a *C-block* of the graph. Leaves of a decomposition tree of any graph correspond to its *C-blocks*. Let  $\mathcal{X}$  be a class of graphs. The set of all graphs whose every *C-block* belongs to  $\mathcal{X}$  will be called the *C-closure of  $\mathcal{X}$*  denoted by  $[\mathcal{X}]_C$ .

**Theorem 1** *If  $\mathcal{X}$  is easy for the coloring problem, then it is so for  $[\mathcal{X}]_C$ .*

**Proof.** Clearly,  $[\mathcal{X}]_C$  is hereditary. All *C-blocks* of a graph  $G \in [\mathcal{X}]_C$  belong to  $\mathcal{X}$ , and the coloring problem can be solved in polynomial time for them. A decomposition tree for  $G$  can be constructed in polynomial time. Hence, by the previous lemma,  $[\mathcal{X}]_C$  is easy for the coloring problem. ■

### 3.2 Modular decomposition and its applications to the weighted coloring problem

A set  $M \subseteq V(G)$  is a *module* in a graph  $G$  if either  $x$  is adjacent to all elements of  $M$  or none of them for each  $x \in V(G) \setminus M$ . Each vertex of  $G$  and the set  $V(G)$  constitute a module called *trivial*. A module  $M$  is a *nontrivial module* in  $G$  if  $|M| > 1$  and  $M \neq V(G)$ . A graph containing no nontrivial modules is said to be *prime*. For instance,  $P_4$  is prime and  $C_4$  does not.

Modular decomposition of graphs is an algorithmic technique based on the following decomposition theorem due to T. Gallai.

**Theorem 2** [7] *Let  $G$  be a graph with at least two vertices. Then exactly one of the following conditions holds:*

- (1).  $G$  is not connected
- (2).  $\overline{G}$  is not connected
- (3).  $G$  and  $\overline{G}$  are connected, and there is a set  $V'$  with at least four elements and an unique partition  $P(G)$  of  $V(G)$  such that
  - (a).  $G(V')$  is a maximal prime induced subgraph of  $G$
  - (b). for each  $V'' \in P(G)$ ,  $V''$  is a module (perhaps, trivial) in  $G$  and  $|V'' \cap V'| = 1$ .

By the theorem, there are decomposition operations of three types. First, if  $G$  is not connected, then disconnect it into connected components  $G_1, \dots, G_p$ . Second, if  $\overline{G}$  has connected components  $\overline{G}_1, \dots, \overline{G}_q$ , then decompose  $G$  into  $G_1, \dots, G_q$ . At length, if  $G$  and  $\overline{G}$  are connected, then its maximal modules are pairwise disjoint, and they form the partition  $P(G)$ . The graph  $G$  is decomposed into subgraphs in  $\{G(V'') \mid V'' \in P(G)\}$ . Additionally, each class of  $P(G)$  is contracted to obtain a graph which is isomorphic to  $G(V')$ . In other words,  $G(V')$  is an induced subgraph of  $G$  producing by taking one element in each class of  $P(G)$ .

The decomposition process above can be represented by an uniquely determined tree called the *modular decomposition tree* of  $G$ . Its vertices are induced subgraphs of  $G$ . A vertex  $G$  has the connected components of  $G$  or  $\overline{G}$  as the children in the first two cases; the children are subgraphs of the form  $G(V'')$ ,  $V'' \in P(G)$  in the third one. Moreover, we associate the graph  $G(V'')$  with the vertex  $G$ . The modular decomposition tree can be determined in  $O(n + m)$ -time for any graph with  $n$  vertices and  $m$  edges [3].

The *weighted coloring problem* is to find, for given  $G$  and a function  $w : V(G) \rightarrow \mathbb{N}$ , the smallest number  $k$  such that there is a function  $c :$

$V(G) \rightarrow 2^{\{1,2,\dots,k\}}$  such that  $|c(v)| = w(v)$  for any  $v$  and  $c(v_1) \cap c(v_2) = \emptyset$  for any adjacent  $v_1$  and  $v_2$ . The elements of  $c(v)$  are called the *colors* of  $v$ . This  $k$  is denoted by  $\chi_w(G)$  and called the *weighted chromatic number* of  $G$ . For every graph  $G$ ,  $\chi_{w'}(G) = \chi(G)$ , where  $w'$  maps every vertex to 1.

Clearly, for each function  $w$ , we have  $\chi_w(G) = \max_i(\chi_w(G_i))$ , where  $G_1, \dots, G_p$  are connected components of  $G$ . Similarly, if  $\overline{G}_1, \dots, \overline{G}_q$  are connected components of  $\overline{G}$ , then  $\chi_w(G) = \sum_{i=1}^q \chi_w(G_i)$ .

**Lemma 2** *Let  $G$  be a graph,  $P(G)$  be its modular decomposition,  $w : V(G) \rightarrow \mathbb{N}$  be an arbitrary function. Then  $\chi_w(G) = \chi_{w^*}(G(V'))$ , where  $w^*(v) = \chi_w(G(V''))$  for each  $v \in V'$ ,  $V'' \in P(G)$ ,  $\{v\} = V' \cap V''$ .*

**Proof.** Contraction of  $V''$  to  $v$  and assignment  $w(v) = \chi_w(G(V''))$  produces a subgraph whose weighted chromatic number is at most  $\chi_w(G)$ . On the other hand, each element of  $N(v)$  cannot have some  $\chi_w(G(V''))$  colors of  $v$ . Hence, the weighted chromatic number of the subgraph is equal to  $\chi_w(G)$ . Therefore,  $\chi_w(G) = \chi_{w^*}(G(V'))$ . ■

Let  $[\mathcal{X}]_P$  be the set of graphs whose every prime induced subgraph belongs to  $\mathcal{X}$ . Clearly,  $[\mathcal{X}]_P$  is hereditary whenever  $\mathcal{X}$  is hereditary. The theorem below follows from the previous lemma and [3].

**Theorem 3** *If  $\mathcal{X}$  is an easy class for the coloring problem, then it is so for  $[\mathcal{X}]_P$ .*

### 3.3 Bipartite Ramsey theorem

A famous Ramsey theorem claims that any graph has a sufficiently large independent set or a sufficiently large clique. There are numerous its analogues for different classes of graphs, e.g. for bipartite graphs. Recall that a graph is *bipartite* if its vertex set can be partitioned into at most two independent sets. These independent sets are called *parts*. A *matching* in a graph is a subset of pairwise nonadjacent edges. The following result is a corollary of theorem 2 from [5] for  $H = K_{s,s}$ .

**Lemma 3** *Any bipartite graph  $G$  having parts  $A$  and  $B$  with  $n > s^{s+1}$  vertices contains subsets  $A' \subseteq A, B' \subseteq B, |A'| = |B'| = \lfloor (\frac{n}{s})^{\frac{1}{s}} \rfloor$  such that  $G(A' \cup B')$  is empty or complete bipartite.*

### 3.4 Connected $\{P_5, K_p - e\}$ -free graphs without clique separators

Let  $G$  be a connected  $\{P_5, K_p - e\}$ -free graph ( $p \geq 3$ ) without clique separators, and let  $Q$  be its maximum clique.

**Lemma 4** *The graph  $G$  is  $O_3$ -free or  $|Q| \leq (p+1)^{p+2}(p-2)$ .*

**Proof.** Assume that  $|Q| > (p+1)^{p+2}(p-2)$ . Let  $N(Q) \triangleq \{y \notin Q \mid \exists x \in Q, (y, x) \in E(G)\}$ . Any element of  $N(Q)$  cannot be adjacent to  $p-2$  or more vertices of  $Q$ . Let us consider a bipartite graph  $G'$  induced by edges between  $Q$  and  $N(Q)$ . As  $G$  has no clique separators,  $Q$  and  $N(Q)$  are parts of  $G'$ . Clearly, the graph  $G'$  has a matching with  $\lfloor \frac{|Q|}{p-2} \rfloor$  edges, and it is  $K_{p-2, p-2}$ -free. Let  $N_1 \triangleq \{u_1, u_2, \dots, u_k\}$  be a maximum subset of  $Q$  such that  $N(Q)$  has vertices  $v_1, v_2, \dots, v_k$  with  $v_i \in N(u_i) \setminus \bigcup_{j \neq i} N(u_j)$  for each  $i$ . By the previous lemma for  $s = p+1$ ,  $k \geq \lfloor \frac{|Q|}{p-2} \rfloor \geq p+1$ . As  $p \geq 3$ ,  $N_2 \triangleq \{v_1, v_2, \dots, v_k\}$  must be an independent set or a clique to avoid an induced  $P_5$ . If  $N_2$  is independent, then there is no a vertex  $v_i$  having a neighbor  $w \notin Q \cup N(Q)$ . Otherwise,  $w$  must be adjacent to all vertices of  $N_2$ , and  $G$  is not  $P_5$ -free. Hence, a possible neighbor  $w \notin Q$  of an element  $v_i \in N_2$  must belong to  $N(Q)$ . To avoid an induced  $P_5$ ,  $w$  must be adjacent to all elements of  $N_2$  or to  $v_i$  only. The second case is realized if and only if  $N_1$  has only one neighbor of  $w$  coinciding with  $u_i$ . In the first case, there are some three non-neighbors  $u_{i_1}, u_{i_2}, u_{i_3}$  of  $w$ , as  $G'$  is  $K_{p-2, p-2}$ -free. But  $v_{i_1}, w, v_{i_2}, u_{i_2}, u_{i_3}$  induce  $P_5$ . Hence, any possible neighbor  $w_i$  of  $v_i$  that lies outside  $Q$  must be adjacent to  $u_i$  and nonadjacent to  $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k$ . Similarly,  $N(w_i) \subseteq N(u_i) \cup \{u_i\}$ . Hence,  $Q$  is a clique separator. Thus,  $N_2$  must be a clique.

Let  $Q'$  be a maximal clique that includes  $N_2$ . Suppose that  $v \in N(Q) \setminus Q'$ . Since  $N_1$  is maximum,  $v$  has neighbors in  $N_1$ , say,  $u_1, \dots, u_q$ . Clearly,  $q \leq p-3$ . To avoid an induced  $P_5$ ,  $v$  must be adjacent to at least  $k-q-1$  vertices among  $v_{q+1}, \dots, v_k$ . Similarly,  $v$  must be adjacent to  $v_1, \dots, v_q$ . Hence,  $v$  is adjacent to at least  $k-1$  vertices of  $N_2$ . To avoid an induced  $K_p - e$ ,  $v \in Q'$ . Thus,  $N(Q) \setminus Q' = \emptyset$ . In fact,  $Q' = N(Q)$  and  $V(G) = Q \cup N(Q)$ , since  $N(Q)$  is a clique separator otherwise. So,  $G$  is  $O_3$ -free. ■

### 3.5 Connected prime $\{P_5, \overline{P_5}\}$ -free graphs

A graph is said to be *perfect* if the clique number and the chromatic number are equal for every its induced subgraph (not necessarily proper). The class of perfect graphs coincides with  $Free(\{C_5, \overline{C_5}, C_7, \overline{C_7}, \dots\})$ , by the strong perfect graph theorem [2].

**Lemma 5** *Any connected prime  $\{P_5, \overline{P_5}\}$ -free graph is perfect or isomorphic to  $C_5$ .*

**Proof.** Every  $\{P_5, \overline{P_5}, C_5\}$ -free graph is perfect, by the strong perfect graph theorem. Let  $G$  be a connected prime  $\{P_5, \overline{P_5}\}$ -graph containing an induced  $C_5$ . Every element of  $V(G) \setminus V(C_5)$  is either adjacent to all vertices of  $C_5$  or to none of them or to two nonadjacent or to three consecutive [6]. Let  $V_i$  be the set of vertices of  $G$  adjacent to the  $(i-1)$ -th and  $(i+1)$ -th vertices of  $C_5$  counting modulo 5. Let  $V_0$  be the set of vertices adjacent to all vertices of  $C_5$ . Any element of  $V_i$  is adjacent to each element of  $V_0 \cup V_{i-1} \cup V_{i+1}$ , nonadjacent to any element of  $V_{i+2} \cup V_{i-2}$ , any element of  $V_i \setminus V(C_5)$  cannot have neighbors outside  $\bigcup_{i=0}^5 V_i \cup V(C_5)$  [6]. Suppose that  $G$  is not isomorphic to  $C_5$ . Then  $V_i$  has at least two elements for some  $i$  or  $V_0 \neq \emptyset$  and  $|V_1| = |V_2| = |V_3| = |V_4| = |V_5| = 1$ . The set  $V_i$  is a nontrivial module in the first case, and  $V(C_5)$  is a nontrivial module in the second one. We have a contradiction with the assumption. ■

## 4 Main result

**Theorem 4** *The class  $Free(\{P_5, \overline{P_5}\})$  and all classes of the form  $Free(\{P_5, K_p - e\})$  are easy for the coloring problem.*

**Proof.** It is known that for any  $P_5$ -free graph  $G$  the inequality  $\chi(G) \leq 4^{w(G)-1}$  holds [9]. Moreover, for each fixed  $k$ , the  $k$ -colorability problem can be solved in polynomial time for  $P_5$ -free graphs [10]. Hence, by these results, Theorem 1 and Lemma 4, the coloring problem for  $\{P_5, K_p - e\}$ -free graphs can be polynomially reduced to the same problem for  $O_3$ -graphs. The coloring problem for  $O_3$ -free graphs is polynomially equivalent to determining the sizes of maximum matchings in the complement graphs. The last problem is known to be polynomial [4]. Hence,  $\{P_5, K_p - e\}$ -free graphs constitute

an easy class for the coloring problem. The class of perfect graphs is easy for the weighted coloring problem [8]. Perfect graphs can be recognized in polynomial time [1]. Hence, by these facts, Theorem 3 and Lemma 5,  $Free(\{P_5, \overline{P_5}\})$  is easy for the coloring problem. ■

## References

- [1] M. Chudnovsky, G. Cornuejols, X. Liu, P. Seymour, K. Vuskovic, *Recognizing berge graphs*, *Combinatorica*, **25**:2 (2005), 143–186.
- [2] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, *The strong perfect graph theorem*, *Annals of Mathematics*, **164**:1 (2006), 51–229.
- [3] A. Cournier, M. Habib, *A new linear algorithm for modular decomposition*, *Lecture Notes in Computer Science*, **787** (1994), 68–84.
- [4] J. Edmonds, *Paths, trees, and flowers*, *Canadian Journal of Mathematics*, **17** (1965), 449–467.
- [5] P. Erdos, A. Hajnal, J. Pach, *Ramsey-type theorem for bipartite graphs*, *Geombinatorics*, **10** (2000), 64–68.
- [6] J. Fouquet, *A decomposition for a class of  $(P_5, \overline{P_5})$ -free graphs*, *Discrete Mathematics*, **121**:1-3 (1993), 75–83.
- [7] T. Gallai, *Transitiv orientierbare graphen*, *Acta Mathematica Academiae Scientiarum Hungaricae*, **18** (1967), 25–66.
- [8] M. Grotschel, L. Lovasz, A. Schrijver, *Polynomial algorithms for perfect graphs*, *Annals of Discrete Mathematics*, **21** (1984), 325–356.
- [9] A. Gyarfás, *Problems from the world surrounding perfect graphs*, *Zastosowania Matematyki Applicationes Mathematicae*, **XIX**:3–4 (1987), 413–441.
- [10] C. Hoang, M. Kaminski, V. Lozin, J. Sawada, X. Shu, *Deciding  $k$ -colorability of  $P_5$ -free graphs in polynomial time*, *Algorithmica* **57** (2010), 74–81.



- [11] D. Kral', J. Kratochvil, Z. Tuza, G. Woeginger, *Complexity of coloring graphs without forbidden induced subgraphs*, Lecture Notes in Computer Science, **2204** (2001), 254–262.
- [12] V. Lozin, D. Malyshev, *Vertex coloring of graphs with few obstructions*, Discrete Applied Mathematics (accepted).
- [13] D. Malyshev, *The coloring problem for classes with two small obstructions*, Optimization Letters, **8**:8 (2014), 2261–2270.
- [14] D. Malyshev, *Two cases of polynomial-time solvability for the coloring problem*, Journal of Combinatorial Optimization, (2015), doi: 10.1007/s10878-014-9792-3.
- [15] R. Tarjan, *Decomposition by clique separators*, Discrete Mathematics, **55**:2 (1985), 221–232.